Then

$$\sqrt{T}\ \mathrm{vec}(\hat{\Phi}_i - \Phi_i) \xrightarrow{d} N(0, G_i \Sigma_{\hat{\alpha}} G_i'), \qquad i = 1, 2, \ldots, \tag{3.7.5}$$

where

$$G_i := \frac{\partial\ \mathrm{vec}(\Phi_i)}{\partial \alpha'} = \sum_{m=0}^{i-1} J(A')^{i-1-m} \otimes \Phi_m.$$

$$\sqrt{T}\ \mathrm{vec}(\hat{\Psi}_n - \Psi_n) \xrightarrow{d} N(0, F_n \Sigma_{\hat{\alpha}} F_n'), \qquad n = 1, 2, \ldots, \tag{3.7.6}$$

where $F_n := G_1 + \cdots + G_n$.

If $(I_K - A_1 - \cdots - A_p)$ is nonsingular,

$$\sqrt{T}\ \mathrm{vec}(\hat{\Psi}_\infty - \Psi_\infty) \xrightarrow{d} N(0, F_\infty \Sigma_{\hat{\alpha}} F_\infty'), \tag{3.7.7}$$

where $F_\infty := \underbrace{(\Psi_\infty', \ldots, \Psi_\infty')}_{p \text{ times}} \otimes \Psi_\infty$.

$$\sqrt{T}\ \mathrm{vec}(\hat{\Theta}_i - \Theta_i) \xrightarrow{d} N(0, C_i \Sigma_{\hat{\alpha}} C_i' + \overline{C}_i \Sigma_{\hat{\sigma}} \overline{C}_i'), \qquad i = 0, 1, 2, \ldots, \tag{3.7.8}$$

where

$$C_0 := 0,\ C_i := (P' \otimes I_K) G_i,\ i = 1, 2, \ldots,\ \overline{C}_i := (I_K \otimes \Phi_i) H,\ i = 0, 1, \ldots,$$

and

$$\begin{aligned} H := \frac{\partial\ \mathrm{vec}(P)}{\partial \sigma'} &= \mathbf{L}_K' \{\mathbf{L}_K [(I_K \otimes P)\mathbf{K}_{KK} + (P \otimes I_K)] \mathbf{L}_K'\}^{-1} \\ &= \mathbf{L}_K' \{\mathbf{L}_K (I_{\mathbf{K}^2} + \mathbf{K}_{KK})(P \otimes I_K)] \mathbf{L}_K'\}^{-1}. \end{aligned}$$

$$\sqrt{T}\ \mathrm{vec}(\hat{\Xi}_n - \Xi_n) \xrightarrow{d} N(0, B_n \Sigma_{\hat{\alpha}} B_n' + \overline{B}_n \Sigma_{\hat{\sigma}} \overline{B}_n'), \tag{3.7.9}$$

where $B_n := (P' \otimes I_K) F_n$ and $\overline{B}_n := (I_K \otimes \Psi_n) H$.

If $(I_K - A_1 - \cdots - A_p)$ is nonsingular,

$$\sqrt{T}\ \mathrm{vec}(\hat{\Xi}_\infty - \Xi_\infty) \xrightarrow{d} N(0, B_\infty \Sigma_{\hat{\alpha}} B_\infty' + \overline{B}_\infty \Sigma_{\hat{\sigma}} \overline{B}_\infty'), \tag{3.7.10}$$

where $B_\infty := (P' \otimes I_K) F_\infty$ and $\overline{B}_\infty := (I_K \otimes \Psi_\infty) H$.

Finally,

$$\begin{aligned} \sqrt{T}(\hat{\omega}_{jk,h} - \omega_{jk,h}) \xrightarrow{d} N(0, d_{jk,h} \Sigma_{\hat{\alpha}} d_{jk,h}' + \overline{d}_{jk,h} \Sigma_{\hat{\sigma}} \overline{d}_{jk,h}') \\ j, k = 1, \ldots, K; h = 1, 2, \ldots, \end{aligned} \tag{3.7.11}$$

where

$$\begin{aligned} d_{jk,h} := \frac{2}{\mathrm{MSE}_j(h)^2} \sum_{i=0}^{h-1} \ & [\mathrm{MSE}_j(h)(e_j' \Phi_i P e_k)(e_k' P' \otimes e_j') G_i \\ & - (e_j' \Phi_i P e_k)^2 \sum_{m=0}^{h-1} (e_j' \Phi_m \Sigma_u \otimes e_j') G_m] \end{aligned}$$

with $G_0 := 0$ and

$$y_t = A_1 y_{t-1} + u_t + M_1 u_{t-1}. \tag{7.2.13}$$

Assuming we have a sample $y_1, \dots, y_T$ generated by this process and defining

$$\mathfrak{A}_p := \begin{bmatrix} I_K & 0 & & \dots & & 0 & 0 \\ -A_1 & I_K & & & & 0 & 0 \\ -A_2 & -A_1 & \ddots & & & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ -A_p & -A_{p-1} & & \ddots & \ddots & 0 & 0 \\ 0 & -A_p & & & \ddots & 0 & 0 \\ \vdots & & \ddots & & & \vdots & \vdots \\ 0 & 0 & & \ddots & & I_K & 0 \\ 0 & 0 & \dots & -A_p & \dots & -A_1 & I_K \end{bmatrix} \tag{7.2.14}$$

we get

$$\mathfrak{A}_1 \begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix} + \begin{bmatrix} -A_1 y_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \overline{\mathfrak{M}}_1 \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_T \end{bmatrix}.$$

Hence, for given, fixed presample values y_0,

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix} \sim N(\mathfrak{A}_1^{-1}\mathbf{y}_0, \mathfrak{A}_1^{-1}\overline{\mathfrak{M}}_1(I_{T+1} \otimes \Sigma_u)\overline{\mathfrak{M}}_1'\mathfrak{A}_1'^{-1}), \tag{7.2.15}$$

where

$$\mathbf{y}_0 := \begin{bmatrix} A_1 y_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The corresponding likelihood function conditional on y_0 is

$$\begin{aligned} & l(A_1, M_1, \Sigma_u | \mathbf{y}, y_0) \\ & \quad \propto |\mathfrak{A}_1^{-1}\overline{\mathfrak{M}}_1(I_{T+1} \otimes \Sigma_u)\overline{\mathfrak{M}}_1'\mathfrak{A}_1'^{-1}|^{-1/2} \\ & \quad\quad \times \exp\{-\tfrac{1}{2}(\mathbf{y} - \mathfrak{A}_1^{-1}\mathbf{y}_0)'\mathfrak{A}_1'[\overline{\mathfrak{M}}_1(I_{T+1} \otimes \Sigma_u)\overline{\mathfrak{M}}_1']^{-1}\mathfrak{A}_1(\mathbf{y} - \mathfrak{A}_1^{-1}\mathbf{y}_0)\} \\ & \quad = |\overline{\mathfrak{M}}_1(I_{T+1} \otimes \Sigma_u)\overline{\mathfrak{M}}_1'|^{-1/2} \exp\{-\tfrac{1}{2}(\mathfrak{A}_1\mathbf{y} - \mathbf{y}_0)'[\overline{\mathfrak{M}}_1(I_{T+1} \otimes \Sigma_u)\overline{\mathfrak{M}}_1']^{-1} \\ & \quad\quad\quad \times (\mathfrak{A}_1\mathbf{y} - \mathbf{y}_0)\}, \end{aligned} \tag{7.2.16}$$

where $|\mathfrak{A}_1| = 1$ has been used.

With the same arguments as in the pure MA case a simple approximation is obtained by setting $u_0 = y_0 = 0$. Then we get